Bielefeld Encounters in
Physics and Mathematics VII

PATH INTEGRALS FROM meV TO MeV

Selected papers from the Symposium
held at the Bielefeld Center for
Interdisciplinary Research
on August 5 — 9, 1985

Edited by

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World Scientific 1986

PATH INTEGRAL ON S^3 AND ITS APPLICATION TO THE ROSEN-MORSE OSCILLATOR

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Abstract

A general prescription of constructing a path integral on S^n is discussed, particularly for $n = 1, 2$ and 3. The path integral on SU(2), expressed in Euler angles, is explicitly calculated. Application is made of the Euler angle path integral to evaluate the propagator for the one-dimensional nonsymmetric Rosen-Morse oscillator by local time rescaling and dimensional extension. The exact energy spectrum and the normalized wave functions of the Rosen-Morse oscillator are obtained.

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I. INTRODUCTION

The polar coordinate path integral has proven useful, but has not received as much attention as it deserves. It is certainly convenient for handling axially or spherically symmetric systems.[1,2] It is also effective in dealing with cases involving winding numbers,[3,4] such as the Aharonov-Bohm effect[5] and polymer entanglements.[6] It becomes even more powerful if it is used with various techniques recently developed for path integration.

Among those new techniques particularly important are the time transformation trick and the dimensional extension prescription. With the aid of a local time rescaling and a dimensional extension from R^3 to R^4, the longstanding hydrogen atom problem has been solved by path integration.[7,8] It is a curious fact that any quadratic propagator can be converted into the free particle propagator by a global space and time transformation.[9]

An advantage of the polar coordinate formulation is, for instance, that the same hydrogen problem can be solved without the dimensional extension.[10] In fact, many of the systems which have the SO(2,1) dynamical symmetry as the hydrogen atom does are exactly soluble by radial path integration if helped by the local time rescaling trick.[11,12] The dimensional extension technique, on the other hand, may be employed in constructing a higher dimensional angular path integral. The one-dimensional Pöschl-Teller oscillator having SO(3) dynamical symmetry is an example, which has been calculated by angular path integration on a two-dimensional sphere.[13]

The purpose of this paper is to show how the path integral on S^3 may be used to solve the one-dimensional nonsymmetric Rosen-Morse oscillator. In Section II, we study path integrals on S^n, particularly for n = 1, 2 and 3. Then, in Section III, we convert the one-dimensional path integral for the Rosen-Morse system into a soluble path integral on S^3 to find the energy spectrum and the normalized wave function.

II. PATH INTEGRALS ON SPHERES

First we consider a general recipe of constructing a path integral

on the n-dimensional sphere S^n imbedded in R^{n+1}. Let us start with the time-sliced representation of Feynman's path integral for a free particle in R^{n+1},

$$K^{(n+1)}(\vec{q}'',\vec{q}';t'',t') = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp[\tfrac{i}{\hbar} S_j] \prod_{j=1}^{N} [\frac{M}{2\pi i\hbar\tau_j}]^{\frac{1}{2}(n+1)} \prod_{j=1}^{N-1} d^{n+1}\vec{q}_j \tag{2.1}$$

where

$$S_j = \tfrac{1}{2}(M/\tau_j)(\Delta\vec{q}_j)^2. \tag{2.2}$$

Note that $\vec{q}_j = \vec{q}(t_j)$, $\Delta q_j = q_j - q_{j-1}$, $\tau_j = t_j - t_{j-1}$, $t' = t_0$ and $t'' = t_N$. The sphere S^n is a coset space SO(n+1)/SO(n), which is most appropriately described by polar coordinates in R^{n+1}. In polar coordinates, the short time action (2.2) takes the form,[1,2]

$$S_j = \tfrac{1}{2}(M/\tau_j)(\Delta q_j)^2 + (M/\tau_j)q_j q_{j-1}(1 - \cos\Theta_j), \tag{2.3}$$

where $q_j = |\vec{q}_j|$, $\hat{q}_j = \vec{q}_j/q_j$ and $\Theta_j = \cos^{-1}(\hat{q}_j \cdot \hat{q}_{j-1})$. The volume element in (2.1) is expressible as

$$\prod_{j=1}^{N-1} d^{n+1}\vec{q}_j = \prod_{j=1}^{N-1} q_j^{\,n}\, dq_j\, d^n\omega_j\,, \tag{2.4}$$

$d^n\omega_j$ being a surface element on the unit sphere.

In order to construct the path integral for a particle freely moving on S^n, we have to take account of the following constraints;

$$q_j = b = \text{constant} \quad \text{for all } j, \tag{2.5}$$

and

$$(\Delta\vec{q}_j)^2 = (\Delta q_j)^2 + q_j^{\,2}\,\Theta_j^{\,2}\,. \tag{2.6}$$

The first constraint is self-evident. The radial variable must be fixed at a value b, the radius of the sphere. The constraint (2.5) can be brought into the path integral (2.1) by setting

$$\exp[(iM/2\hbar\tau_j)(\Delta q_j)^2] = [2\pi i\hbar\tau_j/M]^{\frac{1}{2}}\delta(q_j - q_{j-1}). \tag{2.7}$$

The second constraint may need some clarification. The short time action (2.3) for a free particle in R^{n+1} is based on the Pythagorian relation expressed in polar coordinates,

$$(\Delta\vec{q}_j)^2 = q_j^{\,2} + q_{j-1}^{\,2} - 2q_jq_{j-1}\cos\Theta_j\ , \tag{2.8}$$

which is valid for any finite space separation $\Delta\vec{q}_j = \vec{q}_j - \vec{q}_{j-1}$. On a curved surface, however, the Pythagorian relation holds only locally; $(d\vec{q})^2 = (dq)^2 + q^2\, d\hat{q}\cdot d\hat{q}$. In the discretized version, it should take the form (2.6) which is valid for a small interval. Thus, under the two constraints, the short time action (2.3) must be replaced by

$$\tilde{S}_j = \frac{M}{2\tau_j}(\Delta q_j)^2 + \frac{M}{\tau_j}q_jq_{j-1}(1-\cos\Theta_j) + \frac{1}{4!}(Mq_jq_{j-1}/\tau_j)\,\Theta_j^{\,4}\ . \tag{2.9}$$

In path integration, as is well-known, terms of $O(\tau_j^{\,1+\epsilon})$ if $\epsilon > 0$ can be ignored from the short time action. It is also a standard practice to regard $(\Delta\xi_j)^2$ of any generalized coordinate variable ξ_j as being of order of τ_j. Take the following asymptotic relation,[10]

$$\int_0^c y^{2n}\exp[-\alpha_1y^2 + \alpha_2y^2 + \beta_1y^4 + \beta_2y^4 + O(y^6)]\,dy$$

$$= \int_0^c y^{2n}\exp[-\alpha_1y^2 - \frac{1}{2}\alpha_2\alpha_1^{-1} + \beta_1y^4 + \frac{3}{4}\beta_2\alpha_1^{-2} + O(\alpha_1^{-3})]\,dy \tag{2.10}$$

which is valid for α_1 large, c = constant and n = integer. If we let $\alpha_1 \sim \tau_j^{-1}$ and $y = \Delta\xi_j$, then it is evident $(\Delta\xi_j)^2 \sim \tau_j$. Accordingly,

$$\Theta_j^{\,2}/\tau_j = 2!\,[1-\cos\Theta_j + \frac{1}{4!}\Theta_j^4]/\tau_j \tag{2.11}$$

is a valid approximation in a path integral, which has been used in (2.9). Furthermore, (2.10) again allows us to replace the Θ^4 term in (2.9) by an equivalent one, $n(n-2)\hbar^2\tau_j/(8Mb^2)$. In n dimensions, the ^{4} term

contains n terms of the form $(\Delta\xi)^4/4!$ and $n(n-1)/2$ terms of the form $(\Delta\xi)^2(\Delta\eta)^2/2$. From (2.10), therefore, the above replacement of the Θ^4 term can be justified.

With (2.7), the radial integration of (2.1) is easily performed. As a result,

$$\tilde{K}^{(n)}(\hat{q}'',\hat{q}';t'',t') = \int K^{(n+1)}(q'',\hat{q}'';q',\hat{q}':t'',t')\,q''^n\,dq'' \tag{2.12}$$

yields the path integral for a particle freely moving on S^n,

$$\tilde{K}^{(n)}(\hat{q}'',\hat{q}';t'',t') = \lim_{N\to\infty}\int\prod_{j=1}^{N}\tilde{K}^{(n)}(\hat{q}_j,\hat{q}_{j-1};\tau_j)\prod_{j=1}^{N-1}d^n\omega_j, \tag{2.13}$$

where

$$\tilde{K}^{(n)}(\hat{q}_j,\hat{q}_{j-1};\tau_j) \equiv \tilde{K}_j^{(n)} = [\frac{Mb^2}{2\pi i h\tau_j}]^{n/2}\exp[\frac{i}{\hbar}\tilde{S}_j], \tag{2.14}$$

with

$$\tilde{S}_j = (Mb^2/\tau_j)(1-\cos\Theta_j) + n(n-2)\hbar^2\tau_j/(8Mb^2). \tag{2.15}$$

Employing Gegenbauer's expansion formula,[14]

$$\exp(z\cos\theta) = (2/z)^\nu\,\Gamma(\nu)\sum_{\ell=0}^{\infty}(\ell+\nu)\,C_\ell^\nu(\cos\theta)\,I_{\ell+\nu}(z), \tag{2.16}$$

for $\nu \neq 0,-1,-2,\ldots$ and the asymptotic formula of the Bessel function for z large,[15]

$$I_\lambda(z) \simeq (2\pi z)^{-\frac{1}{2}}\exp[z - (\lambda^2 - \tfrac{1}{4})/(2z)], \tag{2.17}$$

we set $\nu = (n-1)/2$ and $z = (Mb^2/i\hbar\tau_j)$ to find an expression for the short time propagator (2.14),

$$\tilde{K}_j^{(n)} = \tfrac{1}{2}\pi^{-\frac{1}{2}(n+1)}\,\Gamma[\tfrac{1}{2}(n-1)]\sum_{\ell=0}^{\infty}[\ell + \tfrac{1}{2}(n-1)]\,C_\ell^{\frac{1}{2}(n-1)}(\cos\Theta_j)\exp[-\frac{i\hbar\tau_j}{2Mb^2}\ell(\ell+n-1)]. \tag{2.18}$$

The angular integration of (2.13) will lead to the propagator for a particle on S^n,

$$\tilde{K}^{(n)}(\hat{q}'',\hat{q}';\tau) = \tfrac{1}{2}\pi^{-\frac{1}{2}(n+1)}\Gamma[\tfrac{1}{2}(n-1)]\sum_{\ell=0}^{\infty}[\ell+\tfrac{1}{2}(n-1)]\,C_{\ell}^{\frac{1}{2}(n-1)}(\cos\Theta)\exp[-\frac{i\hbar\tau}{2Mb^2}\ell(\ell+n-1)] \qquad (2.19)$$

where $\Theta = \cos^{-1}(\hat{q}'\cdot\hat{q}'')$ and $\tau = t'' - t'$.

In what follows, we examine special cases where $n = 1$, 2 and 3.

i) Path Integrals on S^1

The one-dimensional sphere S^1 is a circle in R^2, which may be viewed as the group manifold of SO(2) or U(1). The circle, if its radius is b, is well represented on polar coordinates $\vec{q} = b(\cos\phi, \sin\phi)$ in R^2. The line element of SO(2) is $b\,d\omega = b\,d\phi$. Thus, the path integral (2.13) becomes

$$\tilde{K}^{(1)}(\phi'',\phi';t'',t') = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp[\frac{i}{\hbar}\tilde{S}_j]\prod_{j=1}^{N}[\frac{Mb^2}{2\pi i\hbar\tau_j}]^{\frac{1}{2}}\prod_{j=1}^{N-1}d\phi_j \qquad (2.20)$$

where

$$\tilde{S}_j = (Mb^2/\tau_j)[1-\cos(\Delta\phi_j)] - (\hbar^2\tau_j/8Mb^2). \qquad (2.21)$$

The short time propagator for this case is given by

$$\tilde{K}_j^{(1)} = (2\pi)^{-1}\sum_{m_j=-\infty}^{\infty}\exp[im_j(\phi_j-\phi_{j-1})]\exp[-(i\hbar\tau_j/2Mb^2)m_j^2]. \qquad (2.22)$$

In converting (2.18) into (2.22), we have utilized the following relations,

$$C_0^0(\cos\theta) = 1, \qquad \lim_{\nu\to 0} n\,\Gamma(\nu)\,C_n^{\nu}(\cos\theta) = 2\cos(n\theta).$$

The integration of (2.20) is now readily carried out, the result being

$$\tilde{K}^{(1)}(\phi'',\phi';\tau) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi''-\phi')}\exp[-\frac{i\hbar\tau}{2Mb^2}m^2], \qquad (2.23)$$

from which read off is the standard energy spectrum, $E_m = (m^2\hbar^2/2Mb^2)$, with $m = 0, 1, 2, \ldots\ldots$

Alternatively, since U(1) is a coset space T(1)/Z where T(1) is the one-dimensional translation group and Z the group of translations by $2\pi n$

(n:integer), we can express the path integral in the T(1)/Z representation. Instead of taking a circle in R^2, we consider a free particle in R^1 for which the short time action (2.2) reads

$$S_j = \tfrac{1}{2}(M/\tau_j)(\Delta x_j)^2,$$

where $x_j \in (-\infty,\infty)$. To realize T(1)/Z, we transform x into $\phi(x) = x/b - 2\pi[x/2\pi b]$, so that we have $\phi(x)\in[0,2\pi)$. Here $[x/2\pi b]$ is Gauss' symbol for the maximum integer $\leq x/2\pi b$. With this transformation, we obtain

$$\tilde{S}_j = \tfrac{1}{2}(Mb^2/\tau_j)(\Delta\phi_j)^2,$$

which can be cast into the form (2.21) and leads us to (2.23). The same result has been obtained previously by using the last action and by summing over winding numbers.[15,16]

(ii) Path integral on S^2

The two-dimensional sphere of radius b can be covered by spherical coordinates in R^3,

$$\vec{q} = b(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

on which (2.7) gives us

$$\tilde{K}^{(2)}(\theta'',\phi'';\theta',\phi';\tau) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp[\frac{i}{\hbar}\tilde{S}_j]\prod_{j=1}^{N}[\frac{Mb^2}{2\pi i\hbar\tau_j}]\prod_{j=1}^{N-1}\sin\theta_j\,d\theta_j\,d\phi_j \qquad (2.24)$$

with

$$\tilde{S}_j = (Mb^2/\tau_j)(1-\cos\Theta_j) \qquad (2.25)$$

where $\cos\Theta_j = \cos\theta_j\cos\theta_{j-1} + \sin\theta_j\sin\theta_{j-1}\cos(\Delta\phi_j)$. The short time propagator on S^2 is

$$\tilde{K}_j^{(2)} = \frac{1}{4\pi}\sum_{\ell_j=0}^{\infty}(2\ell_j+1)\,C_{\ell_j}^{\frac{1}{2}}(\cos\Theta_j)\exp[-(i\hbar\tau_j/2Mb^2)\,\ell_j(\ell_j+1)]. \qquad (2.26)$$

Since

$$C^{\frac{1}{2}}_{\ell}(\cos\theta_j) = P_\ell(\cos\theta_j) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^{m*}_{\ell}(\theta_{j-1},\phi_{j-1})\,Y^{m}_{\ell}(\theta_j,\phi_j),$$

the integration of (2.24) is straightforwardly carried out to yield

$$\tilde{K}^{(2)}(\theta'',\phi'';\theta',\phi';\tau) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^{m*}_{\ell}(\theta',\phi')\,Y^{m}_{\ell}(\theta'',\phi'')\exp[-\frac{i\hbar\tau}{2Mb^2}\ell(\ell+1)]. \qquad (2.27)$$

This coincides with the result[2] obtained for a rigid body having the moment of inertia $I = Mb^2$.

(iii) Path integrals on S^3

A point on the three-dimensional sphere S^3 of radius b can be located by polar coordinates in R^4,

$$\vec{q} = b(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \sin\theta_1\sin\theta_2\sin\theta_3)$$

with $\theta_1\varepsilon[0,\pi)$, $\theta_2\varepsilon[0,\pi)$, $\theta_3\varepsilon[0,2\pi)$. The surface element on S^2 is then given by

$$b^3 d^3\omega = b^3\sin^2\theta_1\sin\theta_2\, d\theta_1\, d\theta_2\, d\theta_3 \qquad (2.28)$$

and the total area of S^3 is $2b^3\pi^2$. The corresponding path integral is

$$\tilde{K}^{(3)}(\theta_1'',\theta_2'',\theta_3'';\theta_1',\theta_2',\theta_3';\tau) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp[\frac{i}{\hbar}\tilde{S}_j]\prod_{j=1}^{N}[\frac{Mb^2}{2\pi i\hbar\tau_j}]^{3/2}\prod_{j=1}^{N-1}d^3\omega_j \qquad (2.29)$$

and the short time propagator is

$$\tilde{K}^{(3)}_j = \frac{1}{2\pi^2}\sum_{\ell_j=0}^{\infty}(\ell_j+1)\,C^1_{\ell_j}(\cos\Theta_j)\exp[-(i\hbar\tau_j/2Mb^2)(\ell_j+\tfrac{1}{2})(\ell_j+\tfrac{3}{2})]. \qquad (2.30)$$

Since S^3 is homeomorphic to the group manifold of SU(2), (2.29) is a path integral on SU(2). In fact, $C^1{}_\ell(\cos\theta)$ in (2.30) is the character of the $(\ell+1)$-dimensional irreducible representation of SU(2).

On the other hand, SU(2) is locally isomorphic to SO(3). An adjoint representation of SU(2) serves as a representation of SO(3). The $(\ell+1)$-dimensional irreducible representation of SU(2), if ℓ is even, provides the singlevalued irreducible representation of SO(3) and, if odd, the doublevalued irreducible representation. In turn, any function defined over SU(2) can be represented over SO(3). As SO(3) is most conveniently described by Euler angles; $\theta\,\varepsilon[0,\pi)$, $\phi\,\varepsilon[0,2\pi)$ and $\psi\,\varepsilon[-2\pi,2\pi)$, the path integral (2.27) may as well be expressed in terms of Euler angles. The sphere S^3 in R^4 can be covered by

$$\vec{q} = b[\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi+\psi),\ \cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi+\psi),\ \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(\phi-\psi),\ \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}(\phi-\psi)].$$

The volume element of SO(3) is $b^3\sin\theta\, d\theta\, d\phi\, d\psi$. The total volume of SO(3) is $16b^3\pi^2$, which is 8 times the total area of S^3. Therefore, for SU(2), the surface element must be given by

$$b^3 d^3\omega = (b^3/8)\sin\theta\, d\theta\, d\phi\, d\psi. \qquad (2.31)$$

In this case, we have also to replace $q''^n dq''$ in (2.6) by $(q''/8)^n dq''$. Thus we find the properly normalized SU(2) path integral in terms of Euler angles,

$$\tilde{K}^{(3)}(\theta'',\phi'',\psi'';\theta',\phi',\psi':\tau)$$

$$= \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp[\frac{i}{\hbar}\tilde{S}_j]\prod_{j=1}^{N}[Mb^2/8\pi i\hbar\tau_j]^{3/2}\prod_{j=1}^{N-1}\sin\theta_j\, d\theta_j\, d\phi_j\, d\psi_j, \qquad (2.32)$$

with

$$\tilde{S}_j = (Mb^2/\tau_j)(1-\cos\tfrac{1}{2}\Omega_j) + 3\hbar^2\tau_j/(8Mb^2). \qquad (2.33)$$

where $\cos\tfrac{1}{2}\Omega_j = \cos\tfrac{1}{2}\theta_j\cos\tfrac{1}{2}\theta_{j-1}\cos\tfrac{1}{2}(\Delta\phi_j+\Delta\psi_j) + \sin\tfrac{1}{2}\theta_j\sin\tfrac{1}{2}\theta_{j-1}\cos\tfrac{1}{2}(\Delta\phi_j-\Delta\psi_j)$.

The corresponding short time propagator is written as

$$\tilde{K}^{(3)}_j = (16\pi^2)^{-1}\sum_{2J_j=0}^{\infty}(2J_j+1)\,C^1_{2J_j}(\cos\tfrac{1}{2}\Omega_j)\exp[-\frac{2i\hbar\tau_j}{Mb^2}J_j(J_j+1)] \qquad (2.34)$$

where we have replaced ℓ_j by $2J_j$ (= 0,1,2,...). Again, $C^1_{2J}(\cos\frac{1}{2}\Omega)$ is the character of the (2J + 1)-dimensional representation of SU(2), which can be expressed by means of Wigner polynomials,[17]

$$C^1_{2J}(\cos\tfrac{1}{2}\Omega_j) = \sin(J+\tfrac{1}{2})\Omega_j/\sin\tfrac{1}{2}\Omega_j$$
$$= \sum_{\mu,\nu=-J}^{J} e^{-i\mu\Delta\phi_j}\, e^{-i\nu\Delta\psi_j}\, P^J_{\mu,\nu}(\cos\theta_j)\, P^{J\,*}_{\mu,\nu}(\cos\theta_{j-1}). \qquad (2.35)$$

Using the orthogonality relation,[18]

$$\int_{-2\pi}^{2\pi}\int_0^{2\pi}\int_0^{\pi} e^{i(\mu'-\mu)\phi}\, e^{i(\nu'-\nu)\psi}\, P^{J\,*}_{\mu,\nu}(\cos\theta)\, P^{J'}_{\mu',\nu'}(\cos\theta)\sin\theta\, d\theta\, d\phi\, d\psi$$
$$= 16\pi^2 (2\ell+1)^{-1}\,\delta_{JJ'}\,\delta_{\mu\mu'}\,\delta_{\nu\nu'}\ , \qquad (2.36)$$

we can easily integrate (2.32) to obtain

$$\tilde{K}^{(3)}(\theta'',\phi'',\psi'';\theta',\phi',\psi';\tau) = \frac{1}{16\pi^2}\sum_{2J=0}^{\infty}(2J+1)C^1_{2J}(\cos\tfrac{1}{2}\Omega)\exp[-\frac{2i\hbar\tau}{Mb^2}J(J+1)] \quad (2.37)$$

where

$$C^1_{2J}(\cos\tfrac{1}{2}\Omega) = \sum_{\mu,\nu=-J}^{J} e^{-i(\phi''-\phi')\mu}\, e^{-i(\psi''-\psi')\nu} P^{J\,*}_{\mu,\nu}(\cos\theta')\, P^J_{\mu,\nu}(\cos\theta''). \qquad (2.38)$$

Here, it is important to notice that (2.37) is not quite correct for the propagator of a free particle on S^3. It does not satisfy the proper boundary condition. The sum over $2J = 0,1,2,\ldots$ may be divided into two parts, one for J = integers and one for J = half-integers, so that

$$\tilde{K}^{(3)}(\theta'',\phi'',\psi'';\theta',\phi',\psi';\tau)$$
$$= \frac{1}{16\pi^2}\Big[\sum_{J=0}^{\infty} + \sum_{J=\frac{1}{2}}^{\infty}\Big](2J+1)\, C^1_{2J}(\cos\tfrac{1}{2}\Omega)\exp[-\frac{2i\hbar\tau}{Mb^2}J(J+1)]. \qquad (2.39)$$

If ϕ'' is replaced by $\phi''+2\pi$, then

$$\tilde{K}^{(3)}(\theta'',\phi''+2\pi,\psi'';\theta',\phi',\psi';\tau)$$
$$= \frac{1}{16\pi^2}\Big[\sum_{J=0}^{\infty} - \sum_{J=\frac{1}{2}}^{\infty}\Big](2J+1)\, C^1_{2J}(\cos\tfrac{1}{2}\Omega)\exp[-\frac{2i\hbar\tau}{Mb^2}J(J+1)]. \qquad (2.40)$$

Thus, we define the following objects,

$$\tilde{K}^{(3)}_{\pm}(\theta'',\phi''+2\pi,\psi'';\theta',\phi',\psi';\tau) = \tfrac{1}{2}[\tilde{K}^{(3)}(\theta'',\phi'',\psi'';\theta',\phi',\psi') \pm \tilde{K}^{(3)}(\theta'',\phi''+2\pi,\psi'';\theta',\phi',\psi')]$$
$$= \frac{1}{16\pi^2}\sum_{J(\pm)}(2J+1)\, C^1_{2J}(\cos\tfrac{1}{2}\Omega)\exp[-\frac{2i\hbar\tau}{Mb^2}J(J+1)], \qquad (2.41)$$

where $J(\pm)$ means that the summation must be carried out over either integers (+) or half-integers (−). Clearly, $\tilde{K}^{(3)}_{+}$ has a period of 2π, belonging to the singlevalued representation of SO(3), whereas $\tilde{K}^{(3)}_{-}$ belongs to the doublevalued representation having a period of 4π.

By setting $I = Mb^2/4$, we get

$$\tilde{K}^{(3)}_{\pm}(\hat{q}'',\hat{q}';\tau) = \frac{1}{16\pi^2\sin\frac{1}{2}\Omega}\sum_{J(\pm)}(2J+1)\sin[(J+\tfrac{1}{2})\Omega]\exp[-\frac{i\hbar\tau}{2I}J(J+1)] \qquad (2.42)$$

which agrees with the standard result for a spherical top except for the additive constant in the energy spectrum.[19,20] In fact, the action (2.33) corresponds to the Lagrangian for a top whose moment of inertia is $I=\frac{1}{4}Mb^2$,

$$L = \tfrac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta). \qquad (2.43)$$

III. THE ROSEN-MORSE OSCILLATOR

Next, we apply the SU(2) path integral (2.32) to the Rosen-Morse oscillator. By the Rosen-Morse oscillator (or the RM oscillator in short), we mean a one-dimensional system bound by a potential of the form,

$$V(x) = A\tanh(ax) - B\,\mathrm{sech}^2(ax), \qquad (3.1)$$

where A, B and a are positive constants. The potential (3.1) was originally introduced by Rosen and Morse to discuss the vibrational states of polyatomic molecules.[21] Recently, it has attracted some attention.[22] For instance, it may be useful as an alternative to the harmonic oscillator potential in calculating quantum mechanical reaction rate constants.[23]

The propagator for the RM oscillator is given by Feynman's path integral,

$$K(x'',x';t'',t') = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp[\tfrac{i}{\hbar} S_j] \prod_{j=1}^{N} [\frac{M}{2\pi i\hbar\tau_j}]^{\frac{1}{2}} \prod_{j=1}^{N-1} dx_j \tag{3.2}$$

where

$$S_j = (M/2\tau_j)(\Delta x_j)^2 - A\tau_j \tanh(ax_j) + B\,\tau_j \operatorname{sech}^2(ax_j), \tag{3.3}$$

which, if integrated, will supply us information concerning the energy spectrum and the corresponding wave functions of the RM oscillator. As is quite apparent from the nontrivial form of the action (3.3), path integration of (3.2) cannot directly be performed. Therefore, we explore alternative sources of information. In fact, as informative as the propagator is the energy-dependent Green's function. Moreover, the latter, too, can be expressed in terms of a path integral. Keeping these in mind, we consider the following path integral,

$$P(x'',x';\tau) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp[\tfrac{i}{\hbar} W_j] \prod_{j=1}^{N} [\frac{M}{2\pi i\hbar\tau_j}]^{\frac{1}{2}} \prod_{j=1}^{N-1} dx_j \tag{3.4}$$

where

$$W_j = (M/2\tau_j)(\Delta x_j)^2 - A\tau_j \tanh(ax_j) + B\tau_j \operatorname{sech}^2(ax_j) + E\tau_j. \tag{3.5}$$

Once (3.4) is path-integrated, we can calculate the Green's function and the propagator, respectively, by

$$G(x'',x';E) = (i\hbar)^{-1} \int P(x'',x';\tau)\, d\tau, \tag{3.6}$$

$$K(x'',x';\tau) = (i/2\pi) \int G(x',x';E) \exp[-\tfrac{i}{\hbar}E(t''-t')]\, dE. \tag{3.7}$$

To evaluate the path integral (3.4), we first transform the variable $x \in (-\infty,\infty)$ into an angular variable $\theta \in (0,\pi)$ by

$$\operatorname{sech} ax = \sin\theta \tag{3.8}$$

where we choose $\theta \gtrless \frac{1}{2}\pi$ for $x \gtrless 0$. The second and the third term of (3.5) can easily be transformed by noting that $\tanh ax = -\cos\theta$ and $\operatorname{sech} ax = \sin\theta$. The transformation of the kinetic term is not simple. Let $g(\theta) = \operatorname{sech}^{-1}(\sin\theta)$. Then we calculate $a\Delta x_j = g(\theta_j) - g(\theta_{j-1})$ by Taylor's expansion about the midpoint $\bar\theta_j = \frac{1}{2}(\theta_j + \theta_{j-1})$. Since $\theta_j = \bar\theta_j + \frac{1}{2}\Delta\theta_j$ and $\theta_{j-1} = \bar\theta_j - \frac{1}{2}\Delta\theta_j$, we obtain

$$a\,\Delta x_j = g'(\bar\theta_j)\Delta\theta_j + g'''(\bar\theta_j)\Delta\theta_j^3/24 + O(\Delta\theta_j^4),$$

where $g'(\bar\theta_j) = \csc\bar\theta_j$ and $g'''(\bar\theta_j) = \csc\bar\theta_j + 2\cos^2\bar\theta_j \csc^3\bar\theta_j$. Noting that $\sin\theta_j \sin\theta_{j-1} = \sin^2\bar\theta_j - \sin^2(\frac{1}{2}\Delta\theta_j)$ and denoting $\sin\theta_j \sin\theta_{j-1}$ by $\tilde{\sin}^2\theta_j$, we write the kinetic term of (3.5) as

$$\frac{M}{2\tau_j}(\Delta x_j)^2 = \frac{4M}{a^2\tau_j \tilde{\sin}^2\theta_j}[\frac{1}{2!}(\tfrac{1}{2}\Delta\theta_j)^2 - \frac{1}{4!}(\tfrac{1}{2}\Delta\theta_j)^4] - \frac{M(3\tilde{\sin}^2\theta_j + 4)}{96a^2\tau_j \tilde{\sin}^4\theta_j}(\Delta\theta_j)^4. \tag{3.9}$$

In the above, as always, we have ignored terms of $O(\tau_j^2)$. According to (2.21), $(\Delta\theta_j)^2 \sim \tau_j$ and hence $(\Delta\theta_j)^4/\tau_j \sim \tau_j$. Furthermore, using (2.21), we replace the last term of (3.9) by $(a^2\hbar^2\tau_j/32M)(3\tilde{\sin}^2\theta_j + 4)$. The approximation (2.22) is again valid here. In addition to the change of variable, we rescale the local time interval τ_j to a new time interval,

$$\sigma_j = \tfrac{1}{4}\tau_j \tilde{\sin}^2\theta_j. \tag{3.10}$$

Putting these considerations together, we express the short time action (3.5) in terms of the new variable θ_j and the new time interval σ_j,

$$W_j = \frac{M}{a^2\sigma_j}[1 - \cos(\tfrac{1}{2}\Delta\theta_j)] + \frac{4A\sigma_j\cos\bar\theta_j}{\tilde{\sin}^2\theta_j} + 4B'\sigma_j + \frac{4E'\sigma_j}{\tilde{\sin}^2\theta_j}, \tag{3.11}$$

where $B' = B + 3a^2\hbar^2/32M$ and $E' = E + a^2\hbar^2/8M$. The measure of (3.4) must also be altered accordingly,

$$\prod_{j=1}^{N} [M/2\pi i\hbar\tau_j]^{\frac{1}{2}} \prod_{j=1}^{N-1} dx_j = (a^2\sin\theta'\sin\theta'')^{\frac{1}{2}} \prod_{j=1}^{N} [M/8\pi i a^2\hbar\sigma_j]^{\frac{1}{2}} \prod_{j=1}^{N-1} d\theta_j . \quad (3.12)$$

The transformed path integral takes the form,

$$P(x'',x';\tau)=(a^2\sin\theta'\sin\theta'')^{\frac{1}{2}}\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp[\tfrac{i}{\hbar}W_j]\prod_{j=1}^{N}[\frac{M}{8\pi i a^2\hbar\sigma_j}]^{\frac{1}{2}}\prod_{j=1}^{N-1}d\theta_j . \quad (3.13)$$

The action (3.11) defined over the new short time interval σ_j can be re-written within the valid approximation,

$$W_j = \frac{M}{a^2\sigma_j}[1-\cos(\tfrac{1}{2}\Delta\theta_j)] + 4B'\sigma_j + \frac{(E'-A)\sigma_j}{\tilde{\cos}^2\frac{1}{2}\theta_j} + \frac{(E'+A)\sigma_j}{\tilde{\sin}^2\frac{1}{2}\theta_j} . \quad (3.14)$$

The next step of our calculation is to convert (3.13) into an SU(2) path integral by introducing two more angular variables. To achieve this, we utilize the asymptotic formula for z large and p an integer,

$$\int_0^{2\pi}\exp[ip\alpha - z(1-\cos\alpha)]\,d\alpha \simeq (2\pi/z)^{\frac{1}{2}}\exp[-(p^2-\tfrac{1}{4})/2z], \quad (3.15)$$

and generate two angles α and β from the last two terms of (3.14) as

$$\exp(\frac{i(E'-A)}{\hbar\tilde{\cos}^2\frac{1}{2}\theta_j}\sigma_j) = (\frac{M\tilde{\cos}^2\frac{1}{2}\theta_j}{2\pi i a^2\hbar\sigma_j})^{\frac{1}{2}}\int_0^{2\pi}\exp[ip\alpha_j + \frac{iM}{a^2\hbar\sigma_j}\tilde{\cos}^2\tfrac{1}{2}\theta_j(1-\cos\alpha_j)]\,d\alpha_j \quad (3.16)$$

$$\exp(\frac{i(E'+A)}{\hbar\tilde{\sin}^2\frac{1}{2}\theta_j}\sigma_j) = (\frac{M\tilde{\sin}^2\frac{1}{2}\theta_j}{2\pi i a^2\hbar\sigma_j})^{\frac{1}{2}}\int_0^{2\pi}\exp[ip\beta_j + \frac{iM}{a^2\hbar\sigma_j}\tilde{\sin}^2\tfrac{1}{2}\theta_j(1-\cos\beta_j)]\,d\beta_j \quad (3.17)$$

where $p = [-2M(E-A)/\hbar^2a^2]^{\frac{1}{2}}$ and $q = [-2M(E+A)/\hbar^2a^2]^{\frac{1}{2}}$ are assumed to be positive integers. Substituting of (3.16) and (3.17) into (3.13) yields

$$P(x'',x';\tau) = a\sin\theta'\sin\theta''\lim_{N\to\infty}\int d\alpha''d\beta''\int\prod_{j=1}^{N}\exp[\tfrac{i}{\hbar}\tilde{W}_j]\prod_{j=1}^{N}[\frac{M}{8\pi i a^2\hbar\sigma_j}]^{3/2}$$
$$\times\prod_{j=1}^{N-1} 4\sin\tfrac{1}{2}\theta_j\cos\tfrac{1}{2}\theta_j\,d\theta_j d\alpha_j d\beta_j \quad (3.18)$$

where

$$\tilde{W}_j = (M/a^2\sigma_j)[1-\cos(\tfrac{1}{2}\Omega_j)] + 4B'\sigma_j + \hbar p\alpha_j + \hbar q\beta_j \quad (3.19)$$

with $\cos\frac{1}{2}\Omega_j = \cos\frac{1}{2}\theta_j\cos\frac{1}{2}\theta_{j-1}\cos\alpha_j + \sin\frac{1}{2}\theta_j\sin\frac{1}{2}\theta_{j-1}\cos\beta_j$. The newly introduced angular variables α_j and β_j may be changed into Euler angles ϕ_j and ψ_j by

$$\alpha_j = \tfrac{1}{2}(\Delta\psi_j + \Delta\phi_j), \qquad \beta_j = \tfrac{1}{2}(\Delta\psi_j - \Delta\phi_j) \quad (3.20)$$

and

$$\int_0^{2\pi} d\alpha_j \int_0^{2\pi} d\beta_j = \tfrac{1}{2}\int_0^{2\pi} d\phi_j \int_{-2\pi}^{2\pi} d\psi_j . \quad (3.21)$$

With (3.20), (3.21) and $\phi' = \psi' = 0$, the path integral (3.18) becomes

$$P(x'',x';\tau) = a\sin\theta'\sin\theta''\exp[4iB'\sigma/\hbar]$$
$$\times\int_0^{2\pi}d\phi''\int_{-2\pi}^{2\pi}d\psi''\exp[\tfrac{1}{2}i(p-q)\phi'' + \tfrac{1}{2}i(p+q)\psi'']\,Q(\theta'',\phi'',\psi'';\theta',0,0;\sigma) \quad (3.22)$$

where

$$\sigma = \tfrac{1}{4}\tau\sin\theta'\sin\theta'' \quad (3.23)$$

and

$$Q(\theta'',\phi'',\psi'';\theta',0,0;\sigma) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp[\tfrac{i}{\hbar}\tilde{S}_j]\prod_{j=1}^{N}[\frac{M}{8\pi i a^2\hbar\sigma_j}]^{3/2}\prod_{j=1}^{N-1}\sin\theta_j d\theta_j d\phi_j d\psi_j \quad (3.24)$$

having

$$\tilde{S}_j = (M/a^2\sigma_j)[\,1 - \cos\tfrac{1}{2}\Omega_j] + 3a^2\hbar^2/(32M). \qquad (3.25)$$

The last path integral (3.24) is identical in form with the SU(2) path integral (2.32). Thus, we have succeeded in reducing the one-dimesional path integral (3.4) for the RM oscillator into an SU(2) path integral. The inverse of the constant a appearing in the potential (3.1) now plays the role of the radius b of S^3. The path integral (2.32) has been evaluated and given by (2.41). If (2.41) is used for (3.24) in (3.22), then

$$P(x'',x';\tau) = a\sin\theta'\sin\theta'' \sum_{J(\pm)}^{\infty} \sum_{\mu,\nu=-J}^{J} \{\frac{2J+1}{16\pi^2}\exp[-\frac{2ia^2\hbar\sigma}{M}J(J+1) + \frac{4iB\sigma}{\hbar}]$$

$$\times P^{J*}_{\mu,\nu}(\cos\theta')P^{J}_{\mu,\nu}(\cos\theta'') \int_0^{2\pi} d\phi'' \int_{-2\pi}^{2\pi} d\psi'' \exp[\tfrac{1}{2}i(p-q-2\mu)\phi'' + \tfrac{1}{2}i(p+q-2\nu)\psi'']\}. \quad (3.26)$$

Recall that $B' = B + 3a^2\hbar^2/32M$. For convenience, let $B = s(s+1)a^2\hbar^2/2M$. Then, completing the angular integrations and the summations over μ and ν, we arrive at

$$P(x',x'';\tau) = a\sin\theta'\sin\theta'' \sum_{J=J_o}^{\infty} (J+\tfrac{1}{2})\exp\{-\frac{2ia^2\hbar\sigma}{M}[J(J+1)-s(s+1)]\}$$

$$\times\ P^{J*}_{\frac{1}{2}(p-q),\frac{1}{2}(p+q)}(\cos\theta')P^{J}_{\frac{1}{2}(p-q),\frac{1}{2}(p+q)}(\cos\theta'') \qquad (3.27)$$

where $J_o = \max\{\tfrac{1}{2}|p-q|,\ \tfrac{1}{2}|p+q|\}$.

At this point, we wish to remark that the short time rescaling (3.10) and the total time rescaling (3.23) are consistent insofar as the time slicing $\tau = \Sigma\tau_j$ is made anisometrically. For details, see ref.24.

Now the Green's function can readily be found via (3.6) in the form,

$$G(x'',x';E) = \frac{2\pi M}{ia\hbar^2} \sum_{J=J_o} \delta(J-s)\, P^{J*}_{\varepsilon+q,\varepsilon}(\tanh ax')P^{J}_{\varepsilon+q,\varepsilon}(\tanh ax'') \qquad (3.28)$$

where $\varepsilon = (p-q)/2$ and we have transformed the θ-variable back into the x-variable by (3.8). In (3.16) and (3.17), p and q have been assumed to be positive integers. Obviously, $|p+q| > |p-q|$, and hence $J_o = \tfrac{1}{2}(p+q)$ which has to be an integer or a half integer. The remaining summation in (3.28) is over either integral or half integral J, depending on the nature of J_o. This means that $n = J - J_o$ is always an integer. Therefore we can shift the summation of (3.28) by letting $J = n + J_o$ as

$$G(x'',x';E) = \frac{2\pi M}{ia\hbar^2} \sum_{n=0}^{\infty} \delta(n+J_o-s)\, P^{s*}_{J_o,\varepsilon}(\tanh ax')\, P^{s}_{J_o,\varepsilon}(\tanh ax''). \qquad (3.29)$$

Here, J_o depends on the energy E via p and q. In general, for a monotonic function f(E), $\delta(f(E)) = \delta(E - E_n)/|f'(E_n)|$ with $f(E_n) = 0$. In (3.29), $f(E) = \tfrac{1}{2}\{[-2M(E-A)/a^2\hbar^2]^{\frac{1}{2}} + [-2M(E+A)/a^2\hbar^2]^{\frac{1}{2}}\} + (n-s)$. Therefore, $f(E_n) = 0$ yields

$$E_n = -\frac{a^2\hbar^2}{2M}(s-n)^2 - \frac{MA^2}{2a^2\hbar^2(s-n)^2} \qquad (3.30)$$

and also $f'(E_n) = (M/a^2\hbar^2)(s-n)/(s-n+\varepsilon)(s-n-\varepsilon)$. As a result, we have

$$(M/a\hbar^2)\,\delta(n+J_o-s) = N_n^2\,\delta(E-E_n) \qquad (3.31)$$

where

$$N_n^2 = a(s-n+\varepsilon)(s-n-\varepsilon)/(s-n). \qquad (3.32)$$

Since $J_o \geq 0$, there are only a limited number of bound states $s \geq n \geq 0$.

Now we are already able to extract the energy values (3.30) from the singularities of the Green's function (3.29). However, in order to find the energy eigenfunction with a correct normalization factor, we go one step further to evaluate the propagator by the Fourier transform (3.7). The calculation of (3.7) is straightforward. The final result is

$$K(x'',x';t'',t') = \sum_{n=0}^{s} [N_n P^{s}_{s-n,\varepsilon}(\tanh ax')]^* [N_n P^{s}_{s-n,\varepsilon}(\tanh ax'')] \exp[-\frac{i}{\hbar}E_n(t''-t')]$$

$$(3.33)$$

where $s = \frac{1}{2}[-1 + (1 + 8MB/a^2\hbar^2)^{\frac{1}{2}}]$ and $\varepsilon = MA/a^2\hbar^2(s-n)$. From (3.33) immediately follows the wave function corresponding to the energy value E_n given by (3.30),

$$\Psi_n(x) = [a(s-n+\varepsilon)(s-n-\varepsilon)/(s-n)]^{\frac{1}{2}} P^{s}_{s-n,\varepsilon}(\tanh ax), \tag{3.34}$$

which is identical with the result[22] obtained by solving Schroedinger's equation. Since the propagator (3.7) satisfies by construction the condition,

$$\lim_{t'\to t''} K(x'',x';t'',t') = \delta(x''-x'), \tag{3.35}$$

it is assured that the wave function (3.34) is correctly normalized.

In the limit $A \to 0$, the energy spectrum (3.30) and the wave function (3.34) reduce to those derived by path integration for the symmetric Rosen-Morse oscillator.[25] The same technique is applicable to the non-symmetric Pöschl-Teller oscillator.[26]

IV. CONCLUDING REMARKS

We have studied a way to construct path integrals on S^n and applied a path integral constructed on S^3 for solving the Rosen-Morse oscillator. Here we wish to make remarks on the following two points.

In constructing the path integral for a particle moving freely on S^n, we have chosen the short time action of the form,

$$S_j = \tfrac{1}{2}(Mb^2/\tau_j)\,\Theta_j^2 \tag{4.1}$$

instead of

$$S_j = \tfrac{1}{2}(Mb^2/\tau_j)\,\Theta_j^2 + O(\Theta_j^4). \tag{4.2}$$

By choosing (4.1), we have intended to reflect into the formulation the Riemannian nature of the spherical surface, that is, the infinitesimal Pythagorian relation defined at each point. By doing so, however, we have

not suggested to ignore the contribution of Θ^4-terms. On the contrary, we have recreated necessary fourth order terms to carry out path integration, replacing (4.1) by

$$S_j = (Mb^2/\tau_j)(1-\cos\Theta_j) + n(n-2)\hbar^2\tau_j/(8Mb^2). \tag{4.3}$$

The second term of (4.3) is the well-known effective potential term. It is usually introduced by hand. Here it follows naturally from (4.1). A short account of the derivation of (4.3) from (4.1) is given in the text. A more detailed version will be given elsewhere.

Secondly, we must comment on the use of the asymptotic formula (2.17) of the modified Bessel function. Recently there has been some skepticism regarding the validity of the formula.[27] The standard asymptotic formula contains a second term which is missing from (2.17). At issue is whether or not the second term is important in a path integral. To our knowledge, Edwards and Gulyaev[1] are the first who employed the formula (2.17), suggesting that it is unimportant. Their calculation involved some errors, but has been substantially improved.[2] The asymptotic formula (2.17), insofar as it is properly used inside a path integral, remains effective, as is seen in our calculation of Sec.II. The justification of the formula has been linked to the justification of Feynman's path integral.[15] The asymptotic relation between the left and right hand sides of (2.17) is valid in a path integral so long as the path integral is accepted as valid. While the term appearing in (2.17) plays a dominant role, the second term in question stays within the reservoir of fluctuation. It is possible to borrow the term from the reservoir for conputational convenience. The skepticism seems to stem from improper applications of the asymptotic formula (2.17) defined only with the background fluctuation. A technical detail of this comment will also be discussed elsewhere.

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